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Robust arithmetic for multivariate Bernstein-form polynomials

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Abstract

There are several ways to represent, to handle and to display curved surfaces in computer-aided geometric design that involve the use of polynomials. This paper deals with polynomials in the Bernstein form. Other work has shown that these polynomials are more numerically stable and robust than power-form polynomials. However, these advantages are lost if conversions to and from the customary power form are made. To avoid this, algebraic manipulations have to be done in the Bernstein basis. Farouki and Rajan (R.T. Farouki, V.T. Rajan, Algorithms for polynomials in Bernstein form, Computer Aided Geometric Design 5 (1988) 1–26) present methods for doing arithmetic on univariate Bernstein-basis polynomials. This paper extends all polynomial arithmetic operations to multivariate Bernstein-form polynomials.

Keywords: Multivariate Bernstein-form polynomials; Geometric modelling; Robustness

1. Introduction

For the representation of curves and surfaces in computer-aided geometric design polynomials are almost always used. For B-rep geometric modellers, the use of parametric polynomials is often the most convenient. However, a more familiar and, in certain respects, simpler representation of a polynomial is the implicit power form. Some set-theoretic geometric modellers make use of this surface representation and employ interval arithmetic to locate an implicit surface in space; this is an approximation to finding roots of the polynomial (see, for example, Bowyer [1] or Snyder [2]).

An implicit power-form polynomial of degree $n \in \mathcal{N}$ in the variable x is defined by:

$$p(x) = \sum_{k=0}^{n} a_k x^k,$$

where $a_k \in \mathcal{R}$. The equation $p(x) = 0$ is the implicit equation corresponding to the polynomial $p(x)$.

An equivalent representation of $p(x)$ can be given in terms of the Bernstein form. For a given $n \in \mathcal{N}$ the Bernstein polynomials of degree n on the unit interval [0,1] are defined by

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad k = 0, 1, \ldots, n \tag{1}$$

and the Bernstein form of the polynomial $p(x)$ is

$$p(x) = \sum_{k=0}^{n} P_k^n B_k^n(x)$$

where P_k^n are the corresponding Bernstein coefficients. The conversion between the power- and Bernstein-form representation is possible and can be performed regardless of the number of variables (see below).

Farouki and Rajan argue in their papers [3,4] that the Bernstein basis is numerically more stable and better conditioned for finding roots than the power form. They also recommend the use of Bernstein polynomials for a stable implementation of geometric modelling algorithms. And, as a consequence of this and a variety of other technical and historical factors, the parametric Bernstein form is widely used in B-rep modellers.

The use of the Bernstein form in set-theoretic geometric modelling requires the manipulation of multivariate Bernstein-form polynomials directly if numerical accuracy and stability are not to be lost. This paper introduces an arithmetic for multivariate polynomials given in this representation and compares the computational load generated by the same operations applied to power- and Bernstein-form polynomials. All the algebra in this paper applies equally to both parametric and implicit polynomials. Our own application (the set-theoretic geometric modeller SvLis [1]) uses the latter, but everything below could just as easily be used for parametric patches and the like.

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2. Bernstein polynomials

The Bernstein polynomials were introduced by S. Bernstein to give a very simple proof of Weierstrass's approximation theorem (see Lorentz [5]). Nowadays they are very popular for generating Bézier, B-spline or NURBS curves and surfaces.

For a given $n \in \mathcal{N}$, the corresponding Bernstein polynomials of degree n in a general interval $[\bar{x}, \bar{x}]$ are defined by

$$B_k^n(x) = \binom{n}{k} \frac{(x-\bar{x})^k (\bar{x}-x)^{n-k}}{(\bar{x}-\bar{x})^n}, \qquad k = 0, 1, \ldots, n \tag{2}$$

Sometimes it is more convenient to consider the Bernstein polynomials in a unit interval [0,1] as the region of interest (see Eq. (1) for their definition). However, this is not a real restriction because a bijection can always be found, which maps the region of interest to the unit interval.

The Bernstein polynomials have many properties that are interesting for the geometric modelling. Lists of them are given by Farouki and Rajan [3], and Spencer [6].

2.1. Conversion between power form and Bernstein form

The purpose of this paper is to present methods that remove the need for power-form to Bernstein-form conversion as much as possible. However, conversion is not always avoidable and often has to be done at least once.

In our earlier report [7] a method is given for finding the Bernstein form of a multivariate polynomial. Another approach is given in the papers written by Garloff [8], and Zettler and Garloff [9].

For this paper we have adopted the way of writing multivariate Bernstein-form polynomials used by Garloff and other authors (e.g. Sherbrooke and Patrikalakis [10]), and this is reproduced here as the notation will be used throughout.

Let $l \in \mathcal{N}$ be the number of variables and $\mathbf{x} = (x_1, \ldots, x_1) \in \mathcal{R}^l$. A multi-index I is defined as $I = (i_1, \ldots, i_l) \in \mathcal{N}^l$. For two given multi-indices $I, J \in \mathcal{N}^l$ we write $I \le J$ if $0 \le i_1 \le j_1, \ldots, 0 \le i_l \le j_l$.

Notation: We set $\mathbf{x}^I$ for a multiplication of $x_1^{i_1} \cdots x_l^{i_1}$.
Notation: The multi-index $\mathbf{0}$ only contains zeros.
Notation: The result of $I + J$ is a multi-index K given by $k_1 = i_1 + j_1, \ldots, k_l = i_l + j_l$.
Notation: The result of $I - J$ is a multi-index K given by $k_1 = i_1 - j_1, \ldots, k_l = i_l - j_l$.
Notation: We write $\binom{I}{J}$ for a multiplication of $\binom{i_1}{j_1} \cdots \binom{i_l}{j_l}$.
Notation: The minimum function min(*I*,*J*) returns a multi-index K by taking $k_1 = \min(i_1, j_1), \ldots, k_l = \min(i_l, j_l)$.
Notation: The maximum function max(*I*,*J*) returns a multi-index K by taking $k_1 = \max(i_1, j_1), \ldots, k_l = \max(i_l, j_l)$.

Let $p(\mathbf{x})$ be a multivariate polynomial in l variables with real coefficients.

Definition. $N = (n_1, \ldots, n_l)$ is the multi-index of maximum degrees so that n_k is the maximum degree of x_k in $p(\mathbf{x})$.

Definition. The set $S = \{I \in N^l : I \le N\}$ contains all the combinations from $\mathcal{R}^l$ which are smaller than or equal to the multi-index N of maximum degree.

Then an arbitrary polynomial $p(\mathbf{x})$ can be written as

$$p(\mathbf{x}) = \sum_{I \in S} \mathbf{a}_I \mathbf{x}^I$$

where $\mathbf{a}_I \in \mathcal{R}$ represents the corresponding coefficient to each $\mathbf{x}^I$. (Note that some of the $\mathbf{a}_I$ may be 0.)

As before a univariate Bernstein polynomial in the variable x of degree n on the unit interval [0,1] is defined by

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

For the multivariate case we consider, without loss of generality,[1] the unit box $U = [0,1]^l$ and the Ith Bernstein polynomial of degree N is defined by

$$B_I^N(\mathbf{x}) = B_{i_1}^{n_1}(x_1) \cdots B_{i_l}^{n_l}(x_l) \qquad \mathbf{x} \in [0,1]^l.$$

The Bernstein coefficients $\mathbf{P}_I(U)$ of $p(\mathbf{x})$ over U are given by

$$\mathbf{P}_I(U) = \sum_{J \le I} \frac{\binom{I}{J}}{\binom{N}{J}} \mathbf{a}_J \qquad I \in S. \tag{3}$$

And so the Bernstein form of a multivariate polynomial $p(\mathbf{x})$ is defined by:

$$p(\mathbf{x}) = \sum_{I \in S} \mathbf{P}_I(U) B_I^N(\mathbf{x}).$$

2.2. Examples

Example 1. A polynomial $pg(x_1, x_2)$ in power form is given by

$$pg(x_1, x_2) = x_1 + x_2 - 1.$$

The maximum degree is $N = (1,1)$ and the set S is

$$S = \{(0,0)(0,1)(1,0)(1,1)\}.$$

The Bernstein coefficients can be calculated by using Eq. (3)

$$b_{(0,0)} = -1 \qquad b_{(0,1)} = 0$$

$$b_{(1,0)} = 0 \qquad \text{and} \qquad b_{(1,1)} = 1.$$

In this case the multivariate Bernstein polynomials are

[1] Because of the bijection mentioned above.

given by

$B^{(11)}_{(00)}(\mathbf{x}) = (1 - x_1)(1 - x_2)$ and

$B^{(11)}_{(01)}(\mathbf{x}) = (1 - x_1)x_2$

$B^{(11)}_{(10)}(\mathbf{x}) = x_1(1 - x_2)$ and $B^{(11)}_{(11)}(\mathbf{x}) = x_1x_2$.

Therefore the Bernstein form $bg(x_1, x_2)$ of the given polynomial $pg(x_1, x_2)$ is

$$bg(x_1, x_2) = -(1 - x_1)(1 - x_2) + x_1x_2.$$

Example 2. A polynomial $pf(x_1,x_2)$ in power form is given by

$$pf(x_1, x_2) = x_1^2x_2 + x_2 + 3.$$

The multi-index N is $N = (2, 1)$ and this yields to the following set S:

$$S = \{(0,0)(0,1)(1,0)(1,1)(2,0)(2,1)\}.$$

Using Eq. (3) gives the Bernstein coefficients:

$b_{(0,0)} = 3 \quad b_{(0,1)} = 4 \quad b_{(1,0)} = 3$

$b_{(1,1)} = 4 \quad b_{(2,0)} = 3$ and $b_{(2,1)} = 5$.

The Bernstein polynomials are given by:

$B^{(21)}_{(00)}(\mathbf{x}) = (1 - x_1)^2(1 - x_2)$ and

$B^{(21)}_{(01)}(\mathbf{x}) = (1 - x_1)^2x_2$

$B^{(21)}_{(10)}(\mathbf{x}) = 2x_1(1 - x_1)(1 - x_2)$ and

$B^{(21)}_{(11)}(\mathbf{x}) = 2x_1(1 - x_1)x_2$

$B^{(21)}_{(20)}(\mathbf{x}) = x_1^2(1 - x_2)$ and $B^{(21)}_{(21)}(\mathbf{x}) = x_1^2x_2$

The Bernstein form $bf(x_1,x_2)$ of the polynomial $pf(x_1,x_2)$ is therefore

$$\begin{aligned} bf(x_1, x_2) = {} & 3(1 - x_1)^2(1 - x_2) \\ & +3(2x_1(1 - x_1)(1 - x_2)) + 3x_1^2(1 - x_2) \\ & +4(1 - x_1)^2x_2 + 4(2x_1(1 - x_1)x_2) + 5x_1^2x_2. \end{aligned}$$

3. Arithmetic for multivariate Bernstein-form polynomials

As mentioned above, if the stability of the Bernstein basis is to be retained while polynomials are being manipulated an arithmetic for Bernstein-form polynomials has to be defined. In their paper [4] Farouki and Rajan give an arithmetic for univariate Bernstein-form polynomials.

In this section an arithmetic for multivariate Bernstein-form polynomials is derived. The resulting arithmetic rules are more complicated than the ones for the univariate case. For the following formulae the way of writing multivariate Bernstein-form polynomials as shown in the last section is adopted.

3.1. Degree elevation

A given multivariate Bernstein-form polynomial $f(\mathbf{x})$ of maximum degree N has a non-trivial representation in a Bernstein basis of higher degree $(N + E)$. The numbers in the multi-index E are equivalent to the times a degree elevation has to be performed for the l variables of $\mathbf{x}$. The new $(N + E)$ Bernstein coefficients $\mathbf{F}_K^{(N+E)}$ can be obtained in the following way:

$$\mathbf{F}_K^{(N+E)} = \sum_{L \in S^*} \frac{\binom{N}{L}\binom{E}{K-L}}{\binom{N+E}{K}} \mathbf{F}_L \qquad K \in S_{\text{new}} \tag{4}$$

where the multi-index $L \in S^* = \{I : I = \max(\mathbf{0}, K - E), \ldots, \min(N, K)\}$ and $K \in S_{\text{new}} = \{I : I = \mathbf{0}, \ldots, (N + E)\}$.

For a bivariate Bernstein-form polynomial $f(x_1,x_2)$ of degree (m,n) this formula can be rewritten in the following manner:

$$\mathbf{F}_{i,j}^{(m+r,n+s)} = \sum_{k=\max(0,i-r)}^{\min(m,i)} \sum_{l=\max(0,j-s)}^{\min(n,j)} \frac{\binom{m}{k}\binom{r}{i-k}}{\binom{m+r}{i}} \frac{\binom{n}{l}\binom{s}{j-l}}{\binom{n+s}{j}} \mathbf{F}_{k,l}$$

where $i = 0, \ldots, m + r$ and $j = 0, \ldots, n + s$. The numbers r and s give how often a degree elevation has to be applied to the variables x_1 and x_2.

3.2. Addition and subtraction

The sum or difference of two multivariate polynomials $f(\mathbf{x})$ and $g(\mathbf{x})$ in Bernstein form can be obtained in a similar way to the univariate case. If both polynomials have the same maximum degree N in $\mathbf{x}$ then the new coefficients of the resulting Bernstein-form polynomial $h(x)$ are given by the sum or difference of the corresponding coefficient sets:

$$\mathbf{H}_K = \mathbf{F}_I \pm \mathbf{G}_J \tag{5}$$

where $\mathbf{F}$ contains the Bernstein coefficients of $f(\mathbf{x})$ and $\mathbf{G}$ of $g(\mathbf{x})$.

If the polynomials do not have the same maximum degree, N degree elevations (see Section 3.1) have to be done beforehand[2] and then Eq. (5) can be used.

[2] Note that it might be necessary to elevate the degrees of both multivariate Bernstein-form polynomials because one may have a higher in, say, x_1, and the other in, say, x_2 at the same time.

3.3. Multiplication

The product of two multivariate Bernstein-form polynomials $f(\mathbf{x})$ with maximum degree N_f and $g(\mathbf{x})$ with maximum degree N_g is a new Bernstein-form polynomial $h(\mathbf{x})$. This new polynomial $h(\mathbf{x})$ has a maximum degree of $N = N_f + N_g$. The Bernstein coefficients $\mathbf{H}_K^{(N)}$ for $h(\mathbf{x})$ can be calculated by

$$\mathrm{H}_K^{(N=N_f+N_g)} = \sum_{L\in S^*} \frac{\binom{N_f}{L}\binom{N_g}{K-L}}{\binom{N_f+N_g}{K}} \mathbf{F}_L^{(N_f)}\mathbf{G}_{K-L}^{(N_g)} \tag{6}$$

where $\mathbf{F}$ and $\mathbf{G}$ contain the Bernstein coefficients of the polynomials $f(\mathbf{x})$ and $g(\mathbf{x})$. The set S^* is given by $S^* = \{I : I = \max(\mathbf{0}, K - N_g), \ldots, \min(N_f, K)\}$ and $K \in S_{new} = \{I : I = 0, \ldots, (N_f + N_g)\}$.

For two bivariate polynomials in Bernstein form $f(x_1, x_2)$ and $g(x_1, x_2)$ this formula can be rewritten as

$$\mathbf{H}_{a,b}^{m+p,n+q} = \sum_{l=\max(0,a-p)}^{\min(m,a)} \sum_{k=\max(0,b-q)}^{\min(n,b)} \times \frac{\binom{m}{l}\binom{p}{a-l}}{\binom{m+p}{a}} \frac{\binom{n}{k}\binom{q}{b-k}}{\binom{n+q}{b}} \mathbf{F}_{l,k}^{m,n}\mathbf{G}_{a-l,b-k}^{p,q}$$

where m, n, p and q are, respectively, the maximum degrees of the polynomials $f(x_1, x_2)$ and $g(x_1, x_2)$.

3.4. Division

Farouki and Rajan [4] showed that the division of two univariate Bernstein-form polynomials leads to a system of equations that has to be solved. The division of two multivariate Bernstein-form polynomials can also be performed by solving a system of equations. However, in this case the system is more complicated than in the univariate case.

If the multivariate Bernstein-form polynomial $f(\mathbf{x})$ is divided by $g(\mathbf{x})$ the quotient and remainder polynomial $q(\mathbf{x})$ and $r(\mathbf{x})$ in Bernstein form have to satisfy following condition:

$$f(\mathbf{x}) = q(\mathbf{x})g(\mathbf{x}) + r(\mathbf{x}). \tag{7}$$

To divide two multivariate Bernstein-form polynomials a main variable has to be chosen first and then the division is performed for this main variable.

Whereas in the univariate case the degrees of the quotient and remainder polynomials are well defined, in the multivariate case the exact degrees of these two polynomials $q(\mathbf{x})$ and $r(\mathbf{x})$ are only well known for the main variable (see Berchtold [11]). However, it is possible to give upper bounds for the degrees of the other variables (see Appendix A). If these bounds are used a division of two multivariate Bernstein-form polynomials can be formulated.

For the polynomials $f(\mathbf{x})$ and $g(\mathbf{x})$ the sets of Bernstein coefficients are given by $\mathbf{F}^{(m,M_2,\ldots,M_l)}$ and $\mathbf{G}^{(n,N_2,\ldots,N_l)}$ where $(m, M_2, \ldots, M_l)$ and $(n, N_2, \ldots, N_l)$ are the maximum degree of the polynomials in $\mathbf{x}$. Let x_1 be the main variable that has a maximum degree of m in the polynomial $f(\mathbf{x})$ and n in the polynomial $g(\mathbf{x})$ and for which the condition $m \geq n$ is satisfied. By using the bounds given the coefficient set for the quotient polynomial is given by $\mathbf{Q}^{(m-n,(m-n)M_2+N_2,\ldots,(m-n)M_l+N_l)}$ and the coefficient set for the remainder polynomial is $\mathbf{R}^{(n-1,(m-n+1)M_2+N_2,\ldots,(m-n+1)M_l+N_l)}$.

The relation in Eq. (7) can be expressed as

$$\begin{aligned}\mathbf{F}^{(m,M_2,\ldots,M_l)} &= \mathbf{Q}^{(m-n,(m-n)M_2+N_2,\ldots,(m-n)M_l+N_l)}\mathbf{G}^{(n,N_2,\ldots,N_l)} \\ &\quad + \mathbf{R}^{(n-1,(m-n+1)M_2+N_2,\ldots,(m-n+1)M_l+N_l)} \\ &= (\mathbf{QG})^{(m,(m-n)M_2+2N_2,\ldots,(m-n)M_l+2N_l)} \\ &\quad + \mathbf{R}^{(n-1,(m-n+1)M_2+N_2,\ldots,(m-n+1)M_l+N_l)}.\end{aligned}$$

As said above, for addition of two multivariate Bernstein-form polynomials it is necessary that the polynomials have the same maximum degree in each variable. For the equation above, this means that for the coefficient set $(\mathbf{QG})$ a $(0, M_2, \ldots, M_l)$-times degree elevation in $\mathbf{x}$ and for $\mathbf{R}$ an $(m - n + 1, N_2, \ldots, N_l)$-times degree elevation in $\mathbf{x}$ has to be performed. Obviously, the sum of the Bernstein coefficient sets $(\mathbf{QG})$ and $\mathbf{R}$ leads to the same degree in the main variable but to a much higher degree in the other variables. Therefore a $(0, (m - n)M_2 + 2N_2, \ldots, (m - n)M_l + 2N_l)$-times degree elevation for the Bernstein coefficient set $\mathbf{F}$ has to be determined, too.

The system of equations for the division of the two multivariate Bernstein-form polynomials can be created by the following relation:

$$\sum_{L_1\in S_1^*} \frac{\binom{D_1}{L_1}\binom{E_1}{K-L_1}}{\binom{D_1+E_1}{K}} \mathbf{F}_{L_1}^{D_1} = \sum_{L_2\in S_2^*} \frac{\binom{D_2}{L_2}\binom{E_2}{K-L_2}}{\binom{D_2+E_2}{K}} (\mathbf{QG})_{L_2}^{D_2} + \sum_{L_3\in S_3^*} \frac{\binom{D_3}{L_3}\binom{E_3}{K-L_3}}{\binom{D_3+E_3}{K}} \mathbf{R}_{L_3}^{D_3} \tag{8}$$

where the multi-index $K \in S_{\text{new}} = \{I : I = (\mathbf{0}, \ldots, (m, \ldots, (m - n + 1)M_l + 2N_l)\}$. The multi-indices $E_1 = (0, (m - n) M_2 + 2N_2, \ldots, (m - n)M_l + 2N_l$, $E_2 = (0, M_2, \ldots, M_l)$ and $E_3 = (m - n + 1, N_2, \ldots, N_l)$ give the degree elevation. The degrees of the coefficient sets are given by the multi-indices $D_1 = (m, M_2, \ldots M_l)$, $D_2 = (m, (m - n)M_2 + 2N_2, \ldots, (m - n) M_l + 2N_l)$ and $D_3 = (n - 1, (m - n + 1)M_2 +$

$N_2, \ldots, (m-n+1)M_l + N_l)$. The three different sets of multi-indices are of the form $S_1^* = \{I : I = \max(\mathbf{0}, K - E_1), \ldots, \min(D_1, K)\}$, $S_2^* = \{I : I = \max(\mathbf{0}, K - E_2), \ldots, \min(D_2, K)\}$ and $S_3^* = \{I : I = \max(\mathbf{0}, K - E_3), \ldots, \min(D_3, K)\}$.

Note that for the multiplication of the coefficient sets **Q** and **G**, Formula 6 for the multiplication of multivariate Bernstein-form polynomials has to be applied

$$(\mathbf{QG})_K^{(m,(m-n)M_2+2N_2,\ldots,(m-n)M_l+2N_l)} = \sum_{L_1 \in S^*} \frac{\binom{D_1}{L}\binom{D_2}{K-L}}{\binom{D_1+D_2}{K}} \mathbf{Q}_L^{D_1} \mathbf{G}_{K-L}^{D_2}$$

where $K \in S_{\mathrm{new}} = \{I : I = (m, (m-n)M_2 + 2N_2, \ldots, (m-n)M_l + 2N_l)\}$. The multi-indices $D_1 = (m-n, (m-n)M_2 + N_2, \ldots, (m-n)M_l + N_l)$ and $D_2 = (n, N_2, \ldots, N_l)$ contain the maximum degree of the polynomials $q(\mathbf{x})$ and $g(\mathbf{x})$.

By using the bounds for the maximum degrees given in Appendix A the coefficient matrix of the quotient $q(x_1, x_2)$ and remainder $r(x_1, x_2)$ have the following form:

$$\mathbf{Q}^{12} = \begin{pmatrix} q_{00} & q_{01} & q_{02} \\ q_{10} & q_{11} & q_{21} \end{pmatrix}$$

and

$$\mathbf{R}^{03} = (\, r_{00} \quad r_{01} \quad r_{02} \quad r_{03} \,).$$

If the Bernstein multiplication for **Q** and **G** is determined, the product (**QG**) has the following initial form:

$$(\mathbf{QG})^{23} = \begin{pmatrix} -q_{00} & -\frac{2}{3}q_{01} & -\frac{1}{3}q_{02} & 0 \\ -\frac{1}{2}q_{10} & \frac{1}{6}q_{00} - \frac{1}{3}q_{11} & \frac{1}{3}q_{01} - \frac{1}{6}q_{21} & \frac{1}{2}q_{02} \\ 0 & \frac{1}{3}q_{10} & \frac{2}{3}q_{11} & q_{21} \end{pmatrix}$$

For this product a degree elevation has to be performed which leads to this matrix:

$$(\mathbf{QG})^{24} = \begin{pmatrix} q_{00} & \frac{1}{4}q_{00} - \frac{1}{2}q_{01} & \frac{1}{3}q_{01} - \frac{1}{6}q_{02} & \frac{1}{4}q_{02} & 0 \\ \frac{1}{2}q_{10} & \frac{1}{8}q_{00} - \frac{1}{8}q_{10} - \frac{1}{4}q_{11} & \frac{1}{12}q_{00} + \frac{1}{6}q_{01} - \frac{1}{6}q_{11} - \frac{1}{12}q_{21} & \frac{1}{4}q_{01} + \frac{1}{8}q_{02} - \frac{1}{8}q_{21} & \frac{1}{2}q_{02} \\ 0 & \frac{1}{4}q_{10} & \frac{1}{6}q_{10} + \frac{1}{3}q_{11} & \frac{1}{2}q_{11} + \frac{1}{4}q_{21} & q_{21} \end{pmatrix}$$

The set S^* is given as $S^* = \{I : I = \max(\mathbf{0}, K - D_2), \ldots, \min(D_1, K)\}$.

3.5. Example

The following example demonstrates the division of the two Bernstein-form polynomials derived in the examples of power-to-Bernstein conversion above. The main variable of the division is x_1. The polynomials $bf(x_1, x_2)$ and $bg(x_1, x_2)$ are given by

$$\begin{aligned} bf(x_1, x_2) = {} & 3(1-x_1)^2(1-x_2) \\ & + 3(2x_1(1-x_1)(1-x_2)) + 3x_1^2(1-x_2) \\ & + 4(1-x_1)^2 x_2 + 4(2x_1(1-x_1)x_2) + 5x_1^2 x_2 \end{aligned}$$

$$bg(x_1, x_2) = -1(1-x_1)(1-x_2) + x_1 x_2.$$

The coefficient matrix of polynomial $bf(x_1, x_2)$ is given by

$$\mathbf{F}^{21} = \begin{pmatrix} 3 & 4 \\ 3 & 4 \\ 3 & 5 \end{pmatrix}$$

and for the polynomial $bg(x_1, x_2)$ the coefficient matrix is

$$\mathbf{G}^{11} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

After an $(m-n+1)$-times degree elevation in the main variable and an N-times degree elevation in the other variable the coefficient matrix **R** has the following form:

$$\mathbf{R}^{24} = \begin{pmatrix} r_{00} & \frac{1}{4}r_{00} + \frac{3}{4}r_{01} & \frac{1}{2}r_{01} + \frac{1}{2}r_{02} & \frac{3}{4}r_{02} + \frac{1}{4}r_{03} & r_{03} \\ r_{00} & \frac{1}{4}r_{00} + \frac{3}{4}r_{01} & \frac{1}{2}r_{01} + \frac{1}{2}r_{02} & \frac{3}{4}r_{02} + \frac{1}{4}r_{03} & r_{03} \\ r_{00} & \frac{1}{4}r_{00} + \frac{3}{4}r_{01} & \frac{1}{2}r_{01} + \frac{1}{2}r_{02} & \frac{3}{4}r_{02} + \frac{1}{4}r_{03} & r_{03} \end{pmatrix}$$

For the matrix **F** an $((m-n)M + N)$-times degree elevation has to be performed which gives

$$\mathbf{F}^{24} = \begin{pmatrix} 3 & \frac{13}{4} & \frac{7}{2} & \frac{15}{4} & 4 \\ 3 & \frac{13}{4} & \frac{7}{2} & \frac{15}{4} & 4 \\ 3 & \frac{7}{2} & 4 & \frac{9}{2} & 5 \end{pmatrix}$$

This leads to the following system of equations which has to be solved

$$3 = -q_{00} + r_{00}$$

$$\frac{13}{4} = -\frac{1}{4}q_{00} - \frac{1}{2}q_{01} + \frac{1}{4}r_{00} + \frac{3}{4}r_{01}$$

$$\frac{7}{2} = -\frac{1}{3}q_{01} - \frac{1}{6}q_{02} + \frac{1}{2}r_{01} + \frac{1}{2}r_{02}$$

$$\frac{15}{4} = -\frac{1}{4}q_{02} + \frac{3}{4}r_{02} + \frac{1}{4}r_{03}$$

$$4 = r_{03}$$

$$3 = -\tfrac{1}{2}q_{10} + r_{00}$$

$$\tfrac{13}{4} = \tfrac{1}{8}q_{00} - \tfrac{1}{8}q_{10} - \tfrac{1}{4}q_{11} + \tfrac{1}{4}r_{00} + \tfrac{3}{4}r_{01}$$

$$\tfrac{7}{2} = \tfrac{1}{12}q_{00} + \tfrac{1}{6}q_{01} - \tfrac{1}{6}q_{11} - \tfrac{1}{12}q_{21} + \tfrac{1}{2}r_{01} + \tfrac{1}{2}r_{02}$$

$$\tfrac{15}{4} = \tfrac{1}{4}q_{01} + \tfrac{1}{8}q_{02} - \tfrac{1}{8}q_{21} - \tfrac{3}{4}r_{02} + \tfrac{1}{4}r_{03}$$

$$4 = \tfrac{1}{2}q_{02} + r_{03}$$

$$3 = r_{00}$$

$$\tfrac{7}{2} = \tfrac{1}{4}q_{10} + \tfrac{1}{4}r_{00} + \tfrac{3}{4}r_{01}$$

$$4 = \tfrac{1}{6}q_{10} + \tfrac{1}{3}q_{11} + \tfrac{1}{2}r_{01} + \tfrac{1}{2}r_{02}$$

$$\tfrac{9}{2} = \tfrac{1}{2}q_{11} + \tfrac{1}{4}q_{21} + \tfrac{3}{4}r_{02} + \tfrac{1}{4}r_{03}$$

$$5 = q_{21} + r_{03}$$

The solution of this system of equations is given by

$$q_{00} = 0,\ \ q_{01} = \tfrac{1}{2},\ \ q_{02} = 0,\ \ q_{10} = 0,\ \ q_{21} = 1,\ \ q_{11} = 1,$$

$$r_{00} = 3,\ \ r_{01} = \tfrac{11}{3},\ \ r_{02} = \tfrac{11}{3},\ \ r_{03} = 4.$$

Therefore the coefficient matrices **Q** and **R** for the quotient $q(x_1, x_2)$ and remainder $r(x_1, x_2)$ are given by

$$\mathbf{Q}^{12} = \begin{pmatrix} 0 & \tfrac{1}{2} & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$\mathbf{R}^{03} = \begin{pmatrix} 3 & \tfrac{11}{3} & \tfrac{11}{3} & 4 \end{pmatrix}.$$

4. Partial derivatives

The derivative of a univariate Bernstein polynomial defined on the unit interval [0,1] is given by

$$\frac{\mathrm{d}}{\mathrm{d}x_1}B_k^n(x_1) = n[B_{k-1}^{n-1}(x_1) - B_k^{n-1}(x_1)], \qquad k = 0, 1, \ldots, n$$

where by convention $B_k^n(x_1) \equiv 0$ if $k < 0$ or $k > n$.

For the Ith multivariate Bernstein polynomial of degree N, which is defined on the unit box $U = [0,1]^l$, the partial derivatives for **x** are obtained by

$$\frac{\partial}{\partial x_1}B_I^N(\mathbf{x}) = n_1[B_{i_1-1}^{n_1-1}(x_1) - B_{i_1}^{n_1-1}(x_1)]\cdots B_{i_l}^{n_l}(x_l),$$

$$\cdots = \cdots \qquad \frac{\partial}{\partial x_l}B_I^N(\mathbf{x}) = B_{i_1}^{n_1}(x_1)\cdots n_l[B_{i_l-1}^{n_l-1}(x_l) - B_{i_l}^{n_l-1}(x_l)],$$

$$\mathbf{x} \in [0,1]^l.$$

where by convention $B_k^n(x_l) \equiv 0$ if $k < 0$ or $k > n$.

5. Computational load

The examples show that for the Bernstein-form polynomials most of the coefficients are non-zero even if most of the coefficients of the equivalent power-form polynomial are zero. This, in general, means that a Bernstein-form polynomial has a larger number of terms (see also our report [7]). In this Section the amount of arithmetic that is involved if the different operations are applied to Bernstein-form and power-form polynomials is compared. The worst case situations are considered. Note that in these cases both representations have the same number of terms and all the coefficients are non-zero.

For the following comparison tables the number of variables in **x** is three as this is the most common requirement for geometric modelling. Since the computational time is almost the same for all the different arithmetic operations no distinction between an addition and multiplication is made for the numbers given in the comparison tables. In all the given formulae a factor calculated from different binomial coefficients is necessary. We assume that a look-up table is used for the calculation of the binomial coefficients (which always involve seven multiplications and one division); this number of operations is not included in the number given in the comparison tables.

Two multivariate polynomials $f(\mathbf{x})$ and $g(\mathbf{x})$ are considered. The maximum degrees of these polynomials are given by $N_f = (n_f^1, n_f^2, n_f^3)$ and $N_g = (n_g^1, n_g^2, n_g^3)$, respectively. The maximum number of coefficients for the two polynomials is $u = (n_f^1 + 1)(n_f^2 + 1)(n_f^3 + 1)$ and $v = (n_g^1 + 1)(n_g^2 + 1)(n_g^3 + 1)$. The multivariate polynomial $h(\mathbf{x})$ is the result if one arithmetic operation is applied to the two polynomials. Obviously, the maximum degree $N_h = (n_h^1, n_h^2, n_h^3)$ of this polynomial $h(\mathbf{x})$ depends on the arithmetic operator applied to $f(\mathbf{x})$ and $g(\mathbf{x})$. For the new polynomial $h(\mathbf{x})$ the number of the terms is given by $w = (n_h^1 + 1)(n_h^2 + 1)(n_h^3 + 1)$.

	Maximum degree of $h(\mathbf{x})$	Number of operations for	
		Bernstein form	Power form
E-times degree elevation	$N_h = N_f + E$	$w(2u-1)$	Does not exist
Addition/Subtraction	$N_h = \max(N_f, N_g)$	$w + w(2u-1) + w(2v-1)$	w
Multiplication	$N_h = N_f + N_g$	$w(3u-1)$	$2uv - 1$

The arithmetic involved in a division is given in a second table. Let $x_1 \in \mathbf{x}$ be the main variable with a maximum degree of n_f^1 and n_g^1 in $f(\mathbf{x})$ and $g(\mathbf{x})$, respectively. A multivariate polynomial for the quotient $q(\mathbf{x})$ with $s = (n_f^1 - n_g^1 + 1)((n_f^1 - n_g^1)n_f^2 + n_g^2 + 1)((n_f^1 - n_g^1)n_f^3 + n_g^3 + 1)$ coefficients and the remainder $r(\mathbf{x})$ with $t = (n_g^1 - 1 + 1)((n_f^1 - n_g^1 + 1)n_f^2 +$

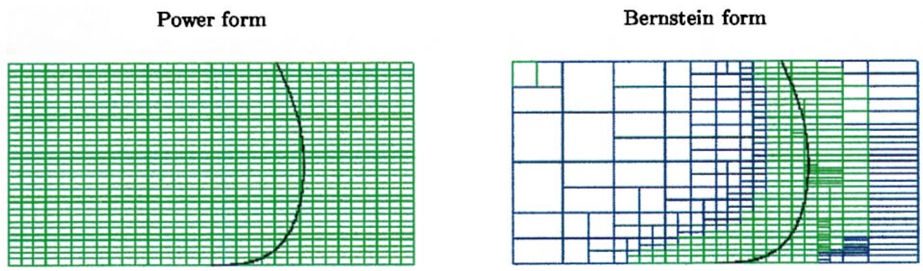

Fig. 1. Recursive division of the modelling volume. On the left—interval arithmetic applied to the power form: all the boxes might contain a part of the surface. On the right—interval arithmetic applied to the Bernstein form: boxes that only take a positive or negative value are detected, along with boxes that might contain a part of the surface.

$n_g^2 + 1)((n_f^1 - n_g^1 + 1)n_f^3 + n_g^3 + 1)$ coefficients[3] is obtained. In this case w is the number of terms obtained by the multiplication of $q(\mathbf{x})$ and $g(\mathbf{x})$. The maximum degrees of the polynomials involved are $(D_1 + E_1)$, $(D_2 + E_2)$ and $(D_3 + E_3)$ and therefore $(d_1 + e_1)$, $(d_2 + e_2)$ and $(d_3 + e_3)$ correspond to the number of coefficients (see also the Section 3.4).

	Number of operations for	
	Bernstein form	Power form
Division	$(d_1 + e_1)(2u - 1)$ $+(d_2 + e_2)(2(w(3v - 1)) - 1)$ $+(d_3 + e_3)(2t - 1)$	$2(n_f^1 - n_g^1 + 1)(n_g^1 + 2)$

The number in the table gives only the arithmetic that will be involved in finding the system of equations. This system can be solved by using Gaussian elimination. In [12] the computational load for Gaussian elimination is given: $((1/3)N^3 + (1/2)N^2M + (1/2)N^2)$ (*one addition + one multiplication*) where N is the number of equations and M the number of unknowns. Therefore another $2((1/3)(d_1 + e_1)^3 + (1/2)(d_1 + e_1)^2(s + t) + (1/2)(d_1 + e_1)^2)$ operations have to be performed to solve the system of equations.

The two tables show that the arithmetic for Bernstein-form polynomials involves much more operations than for the power form. However, to take advantage of the numerical and geometrical robustness of the Bernstein polynomials it is necessary to avoid conversions between the two representations, and this is the price that has to be paid.

6. Conclusion

Section 5 showed that the polynomial arithmetic operations for the Bernstein form involve more computational load. However, if the arithmetic for this representation is not provided, a frequent conversion between the power and the Bernstein basis would be necessary. This, of course, would lead to a loss of the numerical properties and the robustness of the Bernstein form representation. To summarise: more numerical stability and robustness can be obtained by using the Bernstein basis at the cost of a greater computational load.

If these algorithms are provided, the Bernstein form can be employed in set-theoretic geometric modellers. As said in Section 1, these modellers sometimes use interval arithmetic for the location of surfaces in space. In our paper [13] experiments are given, which show that interval arithmetic applied to Bernstein-form polynomials is more accurate for surface location than for the power form.

The following example (see Fig. 1) illustrates the result if interval arithmetic and recursive division are used to locate curves and surfaces in a box-shaped modelling volume. The two methods are applied to a part of a degree-six polynomial given in power form and its equivalent Bernstein form.[4] This polynomial was chosen as being typical of the more complicated shapes our set-theoretic geometric modeller svLis has to represent, such as blend surfaces. The power form of the polynomial has the following equation and is considered in the box $[5, 7] \times [3, 4]$

$$\begin{aligned} 0 = {} & -32556x - 19487y + 3x^2y^4 + 11842xy + 6831y^2 \\ & - 2985x^2y + 3x^4y^2 + 13158x^2 - 3060xy^2 - 60x^3y^2 \\ & + 360x^3y - 18x^4y + 360xy^3 - 30xy^4 + 606x^2y^2 \\ & - 36x^2y^3 + 35592 - 18y^5 - 30x^5 - 1404y^3 - 2980x^3 \\ & + 399x^4 + 207y^4 + x^6 + y^6. \end{aligned}$$

The resulting boxes are labelled depending on their location: a blue box corresponds to a box for which the polynomial only takes positive or negative values and a green box corresponds to a box for which the returned interval straddles zero and therefore the box might contain a part of the surface (that is, the zero of the polynomial).

The picture on the right-hand side of Fig. 1 gives the

[3] The numbers s and t are determined by the bounds for the maximum degree of $q(\mathbf{x})$ and $r(\mathbf{x})$ (see Appendix A).

[4] Note that additional information from the Bernstein form (such as the convex hull property) could be used to improve the box classifications further, but using this would not, of course, give a fair direct comparison.

result if the Bernstein-form polynomial is used. In this case the method returns a number of blue boxes for areas where the polynomial takes only a negative or only a positive value. The green coloured boxes give the region of the modelling space where the surface might be located. For the picture on the left-hand side the equivalent power-form polynomial is used. The method returns only boxes that might contain a part of the surface. Obviously in this example the surface location is more accurate if the Bernstein-form polynomial is used.

The results of the given example and of our previous paper [13] encourage the implementation of inbuilt Bernstein forms into our set-theoretic geometric modeller svLis (which we make available free on the web [1]). The modification, which involves providing an arithmetic for multivariate Bernstein-form polynomials, should automatically improve the accuracy of the location methods and therefore the robustness of the geometric representation itself.

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Appendix A. Appendix

The induction proof of the following theorem was kindly provided by Mulders [14].

Theorem 1. *Let $n \geq m$ be non-negative integers and $x, f_0, f_1, \ldots, f_n, g_0, g_1, \ldots g_m$ be indeterminates. Let $F = f_n x^n + f_{n-1}x^{n-1} + \cdots + f_0$ and $G = g_m x^m + g_{m-1}x^{m-1} + \cdots + g_0$. Then there are polynomials $q, r \in \mathbb{Z}[x, f_0, \ldots, f_n, g_0, \ldots g_m]$ such that*

1. $g_m^{n-m+1}F = qG + r$;
2. $\deg(r, x) < m$;
3. *q is homogeneous in $f_0, \ldots, f_n$ of degree 1*;
4. *q is homogeneous in $g_0, \ldots, g_m$ of degree $n - m$*;
5. *r is homogeneous in $f_0, \ldots, f_n$ of degree 1*;
6. *r is homogeneous in $g_0, \ldots, g_m$ of degree $n - m + 1$*

Proof. By induction to $n - m$. Let

$$\begin{aligned}\tilde{F} &= g_m F + f_n x^{n-m} G \\ &= (g_m f_{n-1} - f_n g_{m-1})x^{n-1} + \cdots + (g_m f_{n-m} - f_n g_0)x^{n-m} \\ &\quad + g_m f_{n-m-1}x^{n-m-1} + \cdots + g_m f_0.\end{aligned}$$

I $n - m = 0$: Take $q = f_n$ and $r = \tilde{F}$.
II $n - m > 0$: Write $\tilde{F} = \tilde{f}_{n-1}x^{n-1} + \cdots + \tilde{f}_0$. Using our induction hypothesis on $\tilde{F}$ and G ($\deg(\tilde{F}, x) - \deg(G) = n - m - 1$) we know that there exist $\tilde{q}, \tilde{r} \in \mathbb{Z}[x, \tilde{f}_0, \cdots \tilde{f}_{n-1}, g_0, \cdots g_m]$ such that

(a) $g_m^{n-m}\tilde{F} = \tilde{q}G + \tilde{r}$;
(b) $\deg(\tilde{r}, x) < m$;
(c) $\tilde{q}$ is homogeneous in $\tilde{f}_0, \ldots, \tilde{f}_{n-1}$ of degree 1;
(d) $\tilde{q}$ is homogeneous in $g_0, \ldots, g_m$ of degree $n - m - 1$;
(e) $\tilde{r}$ is homogeneous in $\tilde{f}_0, \ldots, \tilde{f}_{n-1}$ of degree 1;
(f) $\tilde{r}$ is homogeneous in $g_0, \ldots, g_m$ of degree $n - m$.

Now take $q = g_m^{n-m} f_n x^{n-m} + \tilde{q}$ and $r = \tilde{r}$. Since the $\tilde{f}_i$ are homogeneous of degree 1 in $f_0, \ldots, f_n$ and homogeneous of degree 1 in $g_0, \ldots, g_m$ it now follows that q and r satisfy the conditions.

This proves the theorem. □

Note that for $n < m$ we have for $q = 0$ and $r = F$, that $F = qG + r$ and $\deg(r, x) < \deg(G, x)$.

The division in the theorem is called *pseudo-division*, q is called the *pseudo-quotient* and r the *pseudo-remainder*.

Note that when G is monic, i.e. $g_m = 1$, the division in the theorem is ordinary division.

If now $f_0, \ldots, f_n$ are polynomials in y of degree $\leq N$ and $g_0, \ldots, g_m$ are polynomials in y of degree $\leq M$ we see that q has y-degree $\leq N + (n - m)M$ and r has y-degree $\leq N + (n - m + 1)M$.

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